

# Asymptotic behavior and rigidity results for symmetric solutions of the elliptic system $\Delta u = W_u(u)$

Nicholas D. Alikakos<sup>\*†</sup> and Giorgio Fusco

## Abstract

We study symmetric vector minimizers of the Allen-Cahn energy and establish various results concerning their structure and their asymptotic behavior.

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## 1 Introduction

The problem of describing the structure of bounded solutions  $u : \Omega \rightarrow \mathbb{R}^m$  of the equation

$$(1.1) \quad \begin{cases} \Delta u = f(u), & x \in \Omega \\ u = u_0, & x \in \partial\Omega, \end{cases}$$

where  $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$  is a smooth map and  $\Omega \subset \mathbb{R}^n$  is a smooth domain that can be bounded or unbounded and may also enjoy symmetry properties, is a difficult and important problem which has attracted the interest of many authors in the last twenty five years see [20], [10], [11] and [13] just to mention a few. Questions concerning monotonicity, symmetry and asymptotic behavior are the main objectives of these investigations. Most of the existing literature concerns the scalar case  $m = 1$  where a systematic use of the maximum principle and its consequences are the main tools at hand. For the vector case  $m \geq 2$  we mention the works [12] and [21] where the control of the asymptotic behavior of solutions was basic for proving existence. In this paper we are interested in the case where  $f(u) = W_u(u)$  is the gradient of a potential  $W : \mathbb{R}^m \rightarrow \mathbb{R}$  and  $u$  is a minimizer for the action functional  $\int \frac{1}{2}|\nabla v|^2 + W(v)$  in the sense of the following

**Definition.** A map  $u \in C^2(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)$ ,  $\Omega \subset \mathbb{R}^n$  an open set, is said to be a minimizer or minimal if for each bounded open lipshitz set  $\Omega' \subset \Omega$  it results

$$(1.2) \quad J_{\Omega'}(u) = \min_{v \in W_0^{1,2}(\Omega'; \mathbb{R}^m)} J_{\Omega'}(u + v), \quad J_{\Omega'}(v) = \int_{\Omega'} \frac{1}{2}|\nabla v|^2 + W(v),$$

that is  $u|_{\Omega'}$  is an absolute minimizers in the set of  $W^{1,2}(\Omega'; \mathbb{R}^m)$  maps which coincide with  $u$  on  $\partial\Omega'$ .

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Clearly if  $u : \Omega \rightarrow \mathbb{R}^m$  is minimal then it is a solution of the Euler-Lagrange equation associated to the functional  $J_{\Omega'}$  which is the vector Allen-Cahn equation

$$(1.3) \quad \Delta u = W_u(u), \quad x \in \Omega.$$

We will work in the context of reflection symmetries. Our main results are Theorem 1.2 on the asymptotic behavior of symmetric minimizers and Theorem 1.3 and Theorem 1.5 on the *rigidity* of symmetric minimizers. Rigidity meaning that, under suitable assumptions, a symmetric minimizer  $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$  must in effect depend on a number of variables  $k < n$  strictly less than the dimension  $n$  of the domain space. These theorems, in the symmetric setting, are vector counterparts of analogous results which are well known in the scalar case  $m = 1$  [9] [15]. However in the vector case there is more structure as we explain after the statement of Theorem 1.4. In [8] we discuss a rigidity theorem where the assumption of symmetry is removed.

We let  $G$  a reflection group acting both on the domain space  $\Omega \subseteq \mathbb{R}^n$  and on the target space  $\mathbb{R}^m$ . We assume that  $W : \mathbb{R}^m \rightarrow \mathbb{R}$  a  $C^3$  potential such that

**H<sub>1</sub>**  $W$  is symmetric with respect to  $G$ :  $W(gu) = W(u)$ , for  $g \in G$ ,  $u \in \mathbb{R}^m$ .

For Theorem 1.2 and Theorem 1.3  $G = S$  the group of order 2 generated by the reflection  $\mathbb{R}^d \ni z \mapsto \hat{z} \in \mathbb{R}^d$  in the plane  $\{z_1 = 0\}$ :

$$\hat{z} = (-z_1, z_2, \dots, z_d), \quad d = n, m.$$

In this case the symmetry of  $W$  is expressed by  $W(\hat{u}) = W(u)$ ,  $u \in \mathbb{R}^m$ . For Theorem 1.5  $G = T$  the group of order 6 of the symmetries of the equilateral triangle.  $T$  is generated by the reflection  $\gamma$  in the plane  $\{z_2 = 0\}$  and  $\gamma_{\pm}$  in the plane  $\{z_2 = \pm\sqrt{3}z_1\}$ . We let  $F \subset \mathbb{R}^d$ ,  $d = n$  or  $d = m$  a fundamental region for the action of  $G$  on  $\mathbb{R}^d$ . If  $G = S$  we take  $F = \mathbb{R}_+^d = \{z : z_1 > 0\}$ . If  $G = T$  we take  $F = \{z : 0 < z_2 < \sqrt{3}z_1, z_1 > 0\}$ .

**H<sub>2</sub>** There exists  $a \in \overline{F}$  such that:

$$(1.4) \quad 0 = W(a) \leq W(u), \quad u \in \overline{F}.$$

Moreover  $a$  is nondegenerate in the sense that the quadratic form  $D^2W(a)(z, z)$  is positive definite.

In the symmetric setting we assume minimality in the class of symmetric variations:

**Definition.** Assume that  $\Omega \subset \mathbb{R}^n$  and  $u \in C^2(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)$ , are symmetric

$$(1.5) \quad \begin{aligned} x \in \Omega &\Rightarrow gx \in \Omega, \quad \text{for } g \in G, \\ u(gx) &= gu(x), \quad \text{for } g \in G, x \in \Omega. \end{aligned}$$

Then  $u$  is said to be a symmetric minimizer if for each bounded open symmetric lipschitz set  $\Omega' \subset \Omega$  and for each symmetric  $v \in W_0^{1,2}(\Omega'; \mathbb{R}^m)$  it results

$$(1.6) \quad J_{\Omega'}(u) \leq J_{\Omega'}(u + v).$$

In the following by a minimizer we will always mean a symmetric minimizer in the sense of the definition above.

**Theorem 1.1.** Assume  $G = S$  and assume that  $W$  satisfies  $\mathbf{H}_1 - \mathbf{H}_2$ . Assume that  $\Omega \subseteq \mathbb{R}^n$  is convex-symmetric in the sense that

$$(1.7) \quad x = (x_1, \dots, x_n) \in \Omega \Rightarrow (tx_1, \dots, tx_n) \in \Omega, \text{ for } |t| \leq 1.$$

Let  $\mathcal{Z} = \{z \in \mathbb{R}^m : z \neq a, W(z) = 0\}$  and let  $u : \Omega \rightarrow \mathbb{R}^m$  a minimizer that satisfies

$$(1.8) \quad |u(x) - z| > \delta, \text{ for } z \in \mathcal{Z}, d(x, \partial\Omega^+) \geq d_0, x \in \Omega^+,$$

$\Omega^+ = \{x \in \Omega : x_1 > 0\}$ , and

$$(1.9) \quad |u| + |\nabla u| \leq M, \text{ for } x \in \Omega,$$

for some  $M > 0$

Then there exist  $k_0, K_0 > 0$  such that

$$(1.10) \quad |u - a| \leq K_0 e^{-k_0 d(x, \partial\Omega^+)}, \text{ for } x \in \Omega^+.$$

*Proof.* A minimizer  $u$  satisfies the assumptions of Theorem 1.2 in [18] that implies the result.  $\square$

Examples of minimizers that satisfy the hypothesis of Theorem 1.1 are provided (see [7]) by the entire equivariant solutions of (1.3) constructed in [6], [4], [17]. The gradient bound in (1.9) is a consequence of the smoothness of  $\Omega$  or, as in the case of the entire solutions referred to above, follows from the fact that  $u$  is the restriction to a non smooth set of a smooth map.

We denote  $C_S^{0,1}(\overline{\Omega}, \mathbb{R}^m)$  the set of lipschitz symmetric maps  $v : \overline{\Omega} \rightarrow \mathbb{R}^m$  that satisfy the bounds

$$(1.11) \quad \begin{aligned} \|v\|_{C^{0,1}(\overline{\Omega}, \mathbb{R}^m)} &\leq M, \\ |v - a| + |\nabla v| &\leq K_0 e^{-k_0 d(x, \partial\Omega^+)}, \quad x \in \Omega^+. \end{aligned}$$

We remark that from (1.10) and elliptic regularity, after redefining  $k_0$  and  $K_0$  if necessary, we have

$$(1.12) \quad u \in C_S^{0,1}(\overline{\Omega}, \mathbb{R}^m),$$

for the minimizer in Theorem 1.1.

**Theorem 1.2.** Assume  $W$ ,  $\Omega$  and  $u : \Omega \rightarrow \mathbb{R}^m$  as in Theorem 1.1. Assume moreover that

**H<sub>3</sub>** The problem

$$(1.13) \quad \begin{cases} u'' = W_u(u), & s \in \mathbb{R} \\ u(-s) = \hat{u}(s), & s \in \mathbb{R}, \\ \lim_{s \rightarrow +\infty} u(s) = a, \end{cases}$$

has a unique solution  $\bar{u} : \mathbb{R} \rightarrow \mathbb{R}^m$ .

**H<sub>4</sub>** the operator  $T$  defined by

$$(1.14) \quad D(T) = W_S^{2,2}(\mathbb{R}, \mathbb{R}^m), \quad Tv = -v'' + W_{uu}(\bar{u})v,$$

where  $W_S^{2,2}(\mathbb{R}, \mathbb{R}^m) \subset W^{2,2}(\mathbb{R}, \mathbb{R}^m)$  is the subspace of symmetric maps, has a trivial kernel.

Then there exist  $k, K > 0$  such that

$$(1.15) \quad |u(x) - \bar{u}(x_1)| \leq K e^{-kd(x, \partial\Omega)}, \quad x \in \Omega.$$

**Theorem 1.3.** Assume that  $\Omega = \mathbb{R}^n$  and that  $W$  and  $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$  are as in Theorem 1.2. Then  $u$  is unidimensional:

$$(1.16) \quad u(x) = \bar{u}(x_1), \quad x \in \mathbb{R}^n.$$

**Theorem 1.4.** Assume  $\Omega = \{x \in \mathbb{R}^n : x_n > 0\}$ ,  $W$  and  $u : \Omega \rightarrow \mathbb{R}^m$  as in Theorem 1.2. Then

$$u(x) = \bar{u}(x_1), \quad \text{on } \partial\Omega \Rightarrow u(x) = \bar{u}(x_1), \quad \text{on } \Omega.$$

From [6], [4] and [17], we know that given a finite reflection group  $G$ , provided  $W$  is invariant under  $G$ , there exists a  $G$ -equivariant solutions  $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$  of the system (1.3). It is natural to ask about the asymptotic behavior of these solutions. In particular, given a unit vector  $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{R}^n$  one may wonder about the existence of the limit

$$(1.17) \quad \lim_{\lambda \rightarrow +\infty} u(x' + \lambda\nu) = \tilde{u}(x'),$$

where  $x'$  is the projection of  $x = x' + \lambda\nu$  on the hyperplane orthogonal to  $\nu$ . One can conjecture that this limit does indeed exist and that  $\tilde{u}$  is a solution of the same system equivariant with respect to the subgroup  $G_\nu \subset G$  that leave  $\nu$  fixed, the stabilizer of  $\nu$ . In [6], [4] and [17] an exponential estimate analogous to (1.10) in Theorem 1.1 was established. This gives a positive answer to this conjecture for the case where  $\nu$  is inside the set  $D = \text{Int } \cup_{g \in G_a} g\bar{F}$ . Here  $F$  is a fundamental region for the action of  $G$  on  $\mathbb{R}^d$ ,  $d = n, m$  and  $G_a \subset G$  is the subgroup that leave  $a$  fixed. Under the assumptions **H**<sub>3</sub> and **H**<sub>4</sub> Theorem 1.2 goes one step forward and shows that the conjecture is true when  $\nu$  belongs to the interior of one of the walls of the set  $D$  above and  $G_\nu$  is the subgroup of order two generated by the reflection with respect to that wall. In the proof of Theorem 1.2 the estimate (1.10) is basic. Once the exponential estimate in Theorem 1.2 is established, we conjecture that, under assumptions analogous to **H**<sub>3</sub> and **H**<sub>4</sub>, the approach developed in the proof of Theorem 1.2 can be used to handle the case where  $\nu$  belongs to the intersection of two walls of  $D$ . We also expect that, under the assumption that at each step  $\tilde{u}$  is unique and hyperbolic, the process can be repeated to show the whole hierarchy of limits corresponding to all possible choice of  $\nu$  and always  $\tilde{u}$  is a solution of the system equivariant with respect to the subgroup  $G_\nu$ . This program is motivated by the analogy between equivariant connection maps and minimal cones [5]. Theorem 1.5 below is an example of such a splitting result [24] in the diffused interface set-up. Our next result concerns minimizers equivariant with respect to the symmetry group  $T$  of the equilateral triangle. We can imagine that  $T = G_\nu$  for some  $\nu$  that belongs to the intersection of two walls of  $D$ . The following assumptions **H**<sub>3</sub>' and **H**<sub>4</sub>', in the case at hand  $G = T$ , correspond to the assumption **H**<sub>3</sub> and **H**<sub>4</sub> in Theorem 1.2

**H**<sub>3</sub>' The problem

$$(1.18) \quad \begin{cases} u'' = W_u(u), & s \in \mathbb{R} \\ u(-s) = \gamma u(s), & s \in \mathbb{R}, \\ \lim_{s \rightarrow +\infty} u(s) = \gamma_\pm a, \end{cases}$$

has a unique solution  $\bar{u} : \mathbb{R} \rightarrow \mathbb{R}^m$ .

$\mathbf{H}'_4$  the operator  $T$  defined by

$$(1.19) \quad D(T) = W_{\gamma}^{2,2}(\mathbb{R}, \mathbb{R}^m), \quad Tv = -v'' + W_{uu}(\bar{u})v,$$

where  $W_{\gamma}^{2,2}(\mathbb{R}, \mathbb{R}^m) \subset W^{2,2}(\mathbb{R}, \mathbb{R}^m)$  is the subspace of the maps that satisfy  $u(-s) = \gamma u(s)$ , has a trivial kernel.

Then we have the assumptions concerning uniqueness and hyperbolicity of  $\tilde{u}$

$\mathbf{H}_5$  There is a unique  $G$ -equivariant solution  $\tilde{u} : \mathbb{R}^2 \rightarrow \mathbb{R}^m$  of (1.3)

$$(1.20) \quad \tilde{u}(gs) = g\tilde{u}(s), \quad \text{for } g \in T, s \in \mathbb{R}^2$$

that satisfies the estimate

$$(1.21) \quad |\tilde{u}(s) - a| \leq Ke^{-kd(s, \partial D)}, \quad \text{for } s \in \mathbb{R}^2,$$

where  $D = \text{Int}\bar{F} \cup \gamma\bar{F}$ .

$\mathbf{H}_6$  the operator  $\mathcal{T}$  defined by

$$(1.22) \quad D(\mathcal{T}) = W_G^{2,2}(\mathbb{R}^2, \mathbb{R}^m), \quad \mathcal{T}v = -\Delta v + W_{uu}(\bar{u})v,$$

where  $W_T^{2,2}(\mathbb{R}^2, \mathbb{R}^m) \subset W^{2,2}(\mathbb{R}^2, \mathbb{R}^m)$  is the subspace of  $T$ -equivariant maps, has a trivial kernel.

We are now in the position of stating

**Theorem 1.5.** *Assume that  $W$  satisfies  $\mathbf{H}_1$  and  $\mathbf{H}_2$  with  $a = (1, 0)$  and moreover that  $0 = W(a) < W(u)$  for  $u \in \bar{F}$ . Assume that  $\mathbf{H}'_3$ ,  $\mathbf{H}'_4$  and  $\mathbf{H}_5$ ,  $\mathbf{H}_6$  hold. Let  $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $n \geq 3$  and  $m \geq 2$  be a  $T$ -equivariant minimizer that satisfies (1.9) and, for some  $\delta, d_0 > 0$  the condition*

$$(1.23) \quad |u(x) - \gamma_{\pm}a| \geq \delta \quad \text{for } d(x, \partial D) > d_0, x \in D,$$

where  $D = \{x \in \mathbb{R}^n : |x_2| < \sqrt{3}x_1, x_1 > 0\}$ .

Then  $u$  is two-dimensional:

$$(1.24) \quad u(x) = \tilde{u}(x_1, x_2), \quad x \in \mathbb{R}^n.$$

*Remark.* If instead of a minimizers defined on  $\mathbb{R}^n$  we had considered a minimizer defined on a subset  $\Omega \subset \mathbb{R}^n$ , instead of (1.24), the conclusion of Theorem 1.5 would be exponential convergence of  $u$  to  $\tilde{u}$  similar to (1.15).

Theorem 1.5 is an example of a De Giorgi type result for systems where monotonicity is replaced by minimality ( see [2],[14] and section 3 in [23]). It is the PDE analog of the fact that a minimal cone  $\mathcal{C}$  in  $\mathbb{R}^n$  with the symmetry of the equilateral triangle is necessarily of the form  $\mathcal{C} = \tilde{\mathcal{C}} \times \mathbb{R}^{n-2}$ , with  $\tilde{\mathcal{C}}$  is the triod in the plane. For De Giorgi type results for systems, for general solutions, but under monotonicity hypotheses on the potential  $W$ , we refer to Fazly and Ghoussoub [16]. The rest of the paper is devoted to the proofs. In Section 2 we prove Theorem 1.2 in Section 2.1 and Section 2.2 we prove a number of Lemmas that are basic for the proof of Theorem 1.2 that we conclude in Sections 2.3 and 2.4. Theorems 1.3 and 1.4 and Theorem 1.5 are proved in Section 2.5 and Section 3.

## 2 The proof of Theorem 1.2

The proof of Theorem 1.2 that we present here, from an abstract point of view, has a lot in common with the proof of Theorem 1.2 in [18]. We will remark on this point later and spend a few words to motivate the various lemmas that compose the proof of Theorem 1.2. We begin with some notation and two basic lemmas.

### 2.1 Basic lemmas

In the following we use the notation  $x = (s, \xi)$  with  $x_1 = s$  and  $(x_2, \dots, x_n) = \xi$ . From (1.11) it follows that, if  $(l, \xi) \in \Omega^+$  satisfies  $d((l, \xi), \partial\Omega^+) \geq l$ , then the map  $s \rightarrow u(s, \xi)$ ,  $s \in [-l, l]$ , that we still denote with  $u$  satisfies the bound

$$(2.1) \quad |u - a| + |u_s| \leq K_0 e^{-k_0 s}, \text{ for } s \in [0, l].$$

We denote by  $E_l^{\text{xp}} \subset C^1([-l, l] : \mathbb{R}^m)$  the set of symmetric maps  $v : [-l, l] \rightarrow \mathbb{R}^m$  that satisfy

$$(2.2) \quad |v| + |v_s| \leq K e^{-ks}, \text{ for } s \in [0, l]$$

for some  $k, K > 0$ . We refer to  $E_l^{\text{xp}}$  as the exponential class.

We let  $T_l$  the operator defined by

$$(2.3) \quad D_l(T_l) = \{v \in W_S^{2,2}([-l, l], \mathbb{R}^m) : v(\pm l) = 0\}, \quad T_l v = -v'' + W_{uu}(\bar{u})v.$$

For  $l \in (0, +\infty]$  we let  $\langle v, w \rangle_l = \int_{-l}^l v w$  denote the inner product in  $L^2((-l, l), \mathbb{R}^m)$ . We

let  $\|v\|_l = \langle v, v \rangle_l^{\frac{1}{2}}$  and  $\|v\|_{1,l} = \|v\|_{W^{1,2}([-l, l], \mathbb{R}^m)}$ .

For the standard inner product in  $\mathbb{R}^m$  we use the notation  $(\cdot, \cdot)$ .

It follows directly from (2.2) that  $\|v\|_{1,l} \leq C = \frac{K}{\sqrt{k}}$ . We set

$$(2.4) \quad \mathcal{B}_l^{1,2} := \{v \in W_S^{1,2}([-l, l], \mathbb{R}^m) : v(\pm l) = 0; \|v\|_{1,l} \leq C\},$$

where  $W_S^{1,2}([-l, l], \mathbb{R}^m)$  is the subspace of symmetric maps. Let  $\mathbb{S}$  be defined by

$$(2.5) \quad \mathbb{S} = \{\nu \in W_S^{1,2}([-l, l], \mathbb{R}^m) : \|\nu\|_l = 1\}$$

and set  $q_\nu = \max\{q : q\nu \in \mathcal{B}_l^{1,2}\}$ .

**Lemma 2.1.** *Assume  $H_1$  and  $H_2$  as in Theorem 1.2 and let  $\mathbf{e}_l : \mathcal{B}_l^{1,2} \rightarrow \mathbb{R}$  be defined by*

$$(2.6) \quad \mathbf{e}_l(v) := \frac{1}{2}(\langle \bar{u}_s + v_s, \bar{u}_s + v_s \rangle_l - \langle \bar{u}_s, \bar{u}_s \rangle_l) + \int_{-l}^l (W(\bar{u} + v) - W(\bar{u})).$$

*Then there exist  $l_0 > 0$ ,  $q^\circ > 0$  and  $c > 0$  such that, for all  $l \geq l_0$ , we have*

$$(2.7) \quad \left\{ \begin{array}{ll} D_{qq} \mathbf{e}_l(q\nu) \geq c^2, & \text{for } q \in [0, q^\circ] \cap [0, q_\nu], \nu \in \mathbb{S}, \\ \mathbf{e}_l(q\nu) \geq \mathbf{e}_l(q^\circ \nu), & \text{for } q^\circ \leq q \leq q_\nu, \nu \in \mathbb{S}, \\ \mathbf{e}_l(q\nu) \geq \tilde{\mathbf{e}}_l(p, q, \nu) := \mathbf{e}_l(p\nu) + D_q \mathbf{e}_l(p\nu)(q - p), & \text{for } 0 \leq p < q \leq q_\nu \leq q^\circ, \nu \in \mathbb{S}, \\ D_p \tilde{\mathbf{e}}_l(p, q, \nu) \geq 0, & \text{for } 0 \leq p < q \leq q_\nu \leq q^\circ, \nu \in \mathbb{S}. \end{array} \right.$$

*Remark.*  $\mathbf{e}_l$  is a kind of an *effective* potential. Indeed, as we shall see, in the proof of Theorem 1.2 the map  $L^2((-l, l), \mathbb{R}^m) \ni q \mapsto \mathbf{e}_l(q\nu)$  plays a role similar to the one of the usual potential  $\mathbb{R} \ni q \mapsto W(a + q\nu)$  in the proof of Theorem 1.2 in [18].

*Proof.* By differentiating twice  $\mathbf{e}_l(q\nu)$  with respect to  $q$  gives

$$\begin{aligned} (2.8) \quad D_{qq}\mathbf{e}_l(q\nu) &= \int_{-l}^l (\nu_s, \nu_s) + \int_{-l}^l W_{uu}(\bar{u} + q\nu)(\nu, \nu) \\ &= D_{qq}\mathbf{e}_l(q\nu)|_{q=0} + \int_{-l}^l (W_{uu}(\bar{u} + q\nu) - W_{uu}(\bar{u}))(\nu, \nu). \end{aligned}$$

From the interpolation inequality:

$$\begin{aligned} (2.9) \quad \|v\|_{L^\infty} &\leq \sqrt{2} \|v\|_{1,l}^{\frac{1}{2}} \|v\|_l^{\frac{1}{2}}, \\ &\leq \sqrt{2} \|v\|_{1,l}, \end{aligned}$$

for  $q\nu \in \mathcal{B}_l^{1,2}$  we get via the second inequality

$$(2.10) \quad \|q\nu\|_{L^\infty} \leq \sqrt{2}C,$$

and via the first

$$(2.11) \quad \|\nu\|_{L^\infty} \leq \sqrt{2}C^{\frac{1}{2}}q^{-\frac{1}{2}}.$$

Therefore we have

$$(2.12) \quad |W_{u_i u_j}(\bar{u}(s) + q\nu(s)) - W_{u_i u_j}(\bar{u}(s))| \leq \sqrt{2}C^{\frac{1}{2}}\overline{W}''' q^{\frac{1}{2}},$$

where  $\overline{W}'''$  is defined by

$$(2.13) \quad \overline{W}''' := \max_{\substack{1 \leq i, j, k \leq m \\ s \in \mathbb{R}, |\tau| \leq 1}} W_{u_i u_j u_k}(\bar{u}(s) + \tau\sqrt{2}C).$$

From (2.12) we get

$$(2.14) \quad \left| \int_{-l}^l (W_{uu}(\bar{u} + q\nu) - W_{uu}(\bar{u}))(\nu, \nu) \right| \leq C_1 q^{\frac{1}{2}},$$

where  $C_1 > 0$  is a constant independent of  $l$ . We now observe that

$$(2.15) \quad D_{qq}\mathbf{e}_l(q\nu)|_{q=0} = \langle T_l \nu, \nu \rangle_l = \langle T\tilde{\nu}, \tilde{\nu} \rangle_\infty,$$

where  $\tilde{\nu}$  is the trivial extension of  $\nu$  to  $\mathbb{R}$ .  $T$  is a self-adjoint operator which is positive by the minimality of  $\bar{u}$ . Therefore assumption  $\mathbf{H}_5$  implies that the point spectrum of  $T$  is bounded below by a positive number. From  $\mathbf{H}_2$  the smallest eigenvalue  $\mu$  of the matrix  $W_{uu}(a)$  is positive and Persson's Theorem in [1] implies that also the remaining part of the spectrum of  $T$ , the essential spectrum, is bounded below by  $\mu > 0$ . It follows that the spectrum of  $T$  is bounded below by a positive constant  $0 < \tilde{\mu} \leq \mu$ . From this (2.15) and Theorem 13.31 in [22] it follows

$$(2.16) \quad D_{qq}\mathbf{e}_l(q\nu)|_{q=0} \geq \tilde{\mu},$$

which together with (2.14) implies

$$(2.17) \quad |D_{qq}\mathbf{e}_l(q\nu)| \geq \tilde{\mu} \geq c^2 := \frac{\tilde{\mu}}{2}, \quad \text{for } q \in [0, \bar{q}] \cap [0, q_\nu],$$

where  $\bar{q} = \frac{1}{4} \frac{\tilde{\mu}^2}{C_1}$ . This concludes the proof of (2.7)<sub>1</sub>. We now consider the problem

$$(2.18) \quad \min_{\substack{v \in \mathcal{B}_l^{1,2} \\ \|v\|_l \geq \bar{q}}} \mathbf{e}_l(v)$$

Since the constraint in problem (2.18) is closed with respect to weak convergence in  $W_0^{1,2}$ , if  $\bar{v}_l$  is a minimizer of problem (2.18), we have  $\bar{v}_l \neq 0$ . This implies

$$(2.19) \quad \mathbf{e}_l(\bar{v}_l) = \alpha_l > 0.$$

Indeed the uniqueness assumption about the minimizer  $\bar{u}$  implies that  $v \equiv 0$  is the unique minimizer of  $\mathbf{e}_l$ . We have

$$(2.20) \quad \liminf_{l \rightarrow +\infty} \alpha_l = \alpha > 0.$$

To prove this we assume that instead there is a sequence  $l_k$  such that  $\lim_{k \rightarrow +\infty} \alpha_{l_k} = 0$ . We can also assume that the sequence  $\tilde{v}_{l_k}$  of the trivial extensions of  $\bar{v}_{l_k}$  converges weakly in  $W^{1,2}$  to a map  $\bar{v}$  which by lower semicontinuity satisfies

$$(2.21) \quad \mathbf{e}_\infty(\bar{v}) = 0.$$

This is in contradiction with the assumption that  $v \equiv 0$  is the unique minimizer of  $\mathbf{e}_\infty$  indeed the constraint in problem (2.18) persists in the limit and implies  $\bar{v} \neq 0$ . This establishes (2.20) and concludes the proof of (2.7)<sub>2</sub> with  $q^\circ = \min\{\bar{q}, \alpha\}$ .

The last two inequalities in (2.7) are straightforward consequences of (2.7)<sub>1</sub>.  $\square$

**Lemma 2.2.** *Let  $u$  as in Theorem 1.1 and assume that*

$$(2.22) \quad (l, \xi) \in \Omega^+, \quad d((l, \xi), \partial\Omega^+) \geq l,$$

*then there is a constant  $C_2 > 0$  independent of  $l > 1$ , such that*

$$(2.23) \quad \|u(\cdot, \xi) - \bar{u}\|_{L^\infty([-l, l], \mathbb{R}^m)} \leq C_2 \|u(\cdot, \xi) - \bar{u}\|_l^{\frac{2}{3}}.$$

*Proof.* From (2.22)  $u(\cdot, \xi)$  satisfies (2.1). Since also  $\bar{u}$  satisfies (2.1). There is  $\bar{s} \in [0, l]$  such that  $|u(s, \xi) - \bar{u}(s)| \leq m =: |u(\bar{s}, \xi) - \bar{u}(\bar{s})|$ . From this and  $|u(\cdot, \xi)_s - \bar{u}_s| \leq 2K_0$  it follows

$$(2.24) \quad |u(s, \xi) - \bar{u}(s)| \geq m(1 - 2K_0|s - \bar{s}|), \quad \text{for } s \in [-l, l] \cap [\bar{s} - \frac{m}{2K_0}, \bar{s} + \frac{m}{2K_0}]$$

and a simple computation gives (2.23).  $\square$

Before continuing with the proof, we explain the meaning of the lemmas that follow. Given  $l, r > 0$  and  $\varsigma \in \mathbb{R}^{n-1}$  we let  $\mathcal{C}_l^r(\varsigma) \subset \mathbb{R}^n$  the cylinder

$$(2.25) \quad \mathcal{C}_l^r(\varsigma) := \{(s, \xi) : -l < s < l; |\xi - \varsigma| < r\}.$$



Lemma 2.3, Lemma 2.4 and Lemma 2.5 describe successive deformations through which, fixed  $\lambda > 0$  and  $\varrho > 0$  and  $\bar{q} \in (0, q^\circ)$ , we transform the minimizer  $u$  first into a map  $v$  then into  $w$  and finally into a map  $w^{\bar{q}}$  that satisfies the conditions

$$(2.26) \quad \begin{aligned} w^{\bar{q}} &= u, \quad \text{on } \Omega \setminus \mathcal{C}_{l+\lambda}^{r+2\varrho}(\varsigma), \\ w^{\bar{q}}(l + \frac{\lambda}{2}, \xi) &= \bar{u}(l + \frac{\lambda}{2}), \quad \text{for } |\xi - \varsigma| \leq r + \frac{\varrho}{2}, \\ \|w^{\bar{q}}(\cdot, \xi) - \bar{u}(\cdot)\|_{l+\frac{\lambda}{2}} &\leq \bar{q}, \quad \text{for } |\xi - \varsigma| \leq r + \frac{\varrho}{2} \end{aligned}$$

The deformations described in these lemmas are complemented by precise quantitative estimates on the amount of energy required for the deformation (see (iii) in Lemma 2.3, (iii) in Lemma 2.4 and (2.47) in Lemma 2.5). Lemma 2.3 describes the deformation of  $u$  into a map  $v$  that coincides with  $\bar{u}$  on the lateral boundary of  $\mathcal{C}_{l+\frac{\lambda}{2}}^{r+\varrho}(\varsigma)$ :

$$(2.27) \quad \begin{aligned} v &= u, \quad \text{outside } \mathcal{C}_{l+\lambda}^{r+2\varrho}(\varsigma) \setminus \bar{\mathcal{C}}_l^{r+2\varrho}(\varsigma) \\ \|w(\cdot, \xi) - \bar{u}(\cdot)\|_{l+\frac{\lambda}{2}} &\leq \bar{q}, \quad \text{for } |\xi - \varsigma| = r + \frac{\varrho}{2}. \end{aligned}$$

Lemma 2.4 describes the deformation of  $v$  into a map  $w$  that satisfies

$$(2.28) \quad \begin{aligned} w &= v, \quad \text{outside } \mathcal{C}_{l+\frac{\lambda}{2}}^{r+\varrho}(\varsigma) \setminus \bar{\mathcal{C}}_{l+\frac{\lambda}{2}}^r(\varsigma) \\ \|w(\cdot, \xi) - \bar{u}(\cdot)\|_{l+\frac{\lambda}{2}} &\leq \bar{q}, \quad \text{for } |\xi - \varsigma| = r + \frac{\varrho}{2}. \end{aligned}$$

Lemma 2.6 and Corollary 2.7 show that we can replace  $w^{\bar{q}}$  with a map  $\omega$  that coincides with  $w^{\bar{q}}$  outside  $\mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{\varrho}{2}}(\varsigma)$  and has less energy than  $w^{\bar{q}}$ . Moreover Corollary 2.7 yields a quantitative estimate for the energy difference.

In Sec.2.3 we put together all these energy estimates and show (see Proposition 2.8) that the assumption that

$$\|u(\cdot, \varsigma) - \bar{u}(\cdot)\|_l \geq q^\circ$$

if  $r > 0$  is sufficiently large, is incompatible with the minimality of  $u$ . Thus establishing that, if a sufficiently large cylinder  $\mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{\varrho}{2}}(\varsigma)$  is contained in  $\Omega$ , then we have the estimate

$$\|u(\cdot, \varsigma) - \bar{u}(\cdot)\|_l < q^\circ,$$

which is the main step in the proof of Theorem 1.2.

## 2.2 Replacement Lemmas

**Lemma 2.3.** *Let  $\lambda$  and  $\varrho > 0$  be fixed. Assume that  $\mathcal{C}_{l+\lambda}^{r+2\varrho}(\varsigma) \subset \Omega$  satisfies*

$$(2.29) \quad d(\mathcal{C}_{l+\lambda}^{r+2\varrho}(\varsigma), \partial\Omega) \geq l + \lambda.$$

*Then there exists a map  $v \in C_S^{0,1}(\bar{\Omega}, \mathbb{R}^m)$  such that*

- (i)  $v = u$ , on  $\bar{\Omega} \setminus (\mathcal{C}_{l+\lambda}^{r+2\varrho}(\varsigma) \setminus \bar{\mathcal{C}}_l^{r+2\varrho}(\varsigma))$ ,
- (ii)  $v(l + \frac{\lambda}{2}, \xi) = \bar{u}(l + \frac{\lambda}{2})$ , for  $|\xi - \varsigma| \leq r + \varrho$ .

$$(iii) \quad J_{\mathcal{C}_{l+\lambda}^{r+2\varrho}(\varsigma)}(v) - J_{\mathcal{C}_l^{r+2\varrho}(\varsigma)}(u) \leq C_0 r^{n-1} e^{-2kl},$$

where  $C_0 > 0$  is a constant independent of  $l$  and  $r$ .

*Proof.* For  $(s, \xi) \in \overline{\mathcal{C}}_{l+\lambda}^{r+2\varrho}(\varsigma) \setminus \mathcal{C}_l^{r+2\varrho}(\varsigma)$  we define  $v$  by

$$(2.30) \quad v(s, \xi) = (1 - |1 - 2\frac{s-l}{\lambda}|)\bar{u}(s) + |1 - 2\frac{s-l}{\lambda}|u(s, \xi),$$

$$s \in [l, l + \lambda], \quad |\xi - \varsigma| \leq r + \varrho.$$

It remains to define  $v(s, \xi)$  for  $(s, \xi) \in (l, l + \lambda) \times \{\xi : r + \varrho < |\xi - \varsigma| < r + 2\varrho\}$ .

Set

$$(2.31) \quad Bu(s, \xi) = |\frac{s-l-\lambda}{\lambda}|u(l, \xi) + \frac{s-l}{\lambda}u(l + \lambda, \xi),$$

$$\tilde{u}(s, \xi) = u(s, \xi) - Bu(s, \xi).$$

Note that by (2.30)  $|\xi - \varsigma| = r + \varrho$  implies  $v(l, \xi) = u(l, \xi)$ ,  $v(l + \lambda, \xi) = u(l + \lambda, \xi)$  and therefore we have

$$(2.32) \quad |\xi - \varsigma| = r + \varrho \Rightarrow Bu(s, \xi) = Bv(s, \xi),$$

where  $v$  is defined in (2.30). Set

$$(2.33) \quad \hat{v}(s, \xi) = v(s, (r + \varrho)\frac{\xi - \varsigma}{|\xi - \varsigma|} + \varsigma) - Bu(s, (r + \varrho)\frac{\xi - \varsigma}{|\xi - \varsigma|} + \varsigma),$$

where again  $v$  is defined in (2.30). With these notations we complete the definition of  $v$  by setting

$$(2.34) \quad v(s, \xi) = Bu(s, \xi) + \frac{|\xi - \varsigma| - r - \varrho}{\varrho}\tilde{u}(s, \xi) + \frac{2\varrho + r - |\xi - \varsigma|}{\varrho}\hat{v}(s, \xi),$$

$$\text{for } (s, \xi) \in (l, l + \lambda) \times \{\xi : r + \varrho < |\xi - \varsigma| < r + 2\varrho\}.$$

Statement (i) and (ii) are obvious consequences of the definition of  $v$ . Direct inspection of (2.30) and (2.34) shows that  $v$  is continuous. From (2.30)  $v(s, \xi)$  is a linear combination of  $\bar{u}(s)$  and  $u(s, \xi)$  computed for  $s \in [l, l + \lambda]$ . A similar statement applies to  $v(s, \xi)$  in (2.34) since  $Bu(s, \xi)$ ,  $\hat{v}(s, \xi)$  and  $\tilde{u}(s, \xi)$  are linear combinations of  $u(s, \xi)$  and  $v(s, \xi)$  in (2.30) computed for  $s \in [l, l + \lambda]$ . From this, assumption (2.29) and (2.1) we conclude

$$(2.35) \quad |v - a| + |\nabla v| \leq C_3 e^{-k_0 l} \quad \text{for } (s, \xi) \in \mathcal{C}_{l+\lambda}^{r+2\varrho}(\varsigma) \setminus \overline{\mathcal{C}}_l^{r+2\varrho}(\varsigma),$$

where  $C_3 > 0$  is a constant independent of  $l$  and  $r$ . From (2.35) and the assumptions on the potential  $W$  it follows

$$(2.36) \quad \frac{1}{2}|\nabla v|^2 + W(v) \leq C_4 e^{-2k_0 l},$$

which together with  $\mathcal{H}^n(\mathcal{C}_{l+\lambda}^{r+2\varrho}(\varsigma) \setminus \overline{\mathcal{C}}_l^{r+2\varrho}(\varsigma)) \leq C_5 r^{n-1}$  concludes the proof.  $\square$

Given a number  $0 < \bar{q} < q^\circ$ , let  $A_{\bar{q}}$  be the set

$$(2.37) \quad A_{\bar{q}} := \{\xi : \|v(\cdot, \xi) - \bar{u}(\cdot)\|_{l+\frac{\lambda}{2}} > \bar{q}, \quad |\xi - \varsigma| < r + \varrho\},$$

where  $v$  is the map constructed in Lemma 2.3.

**Lemma 2.4.** *Let  $v$  as before and let  $S := A_{\bar{q}} \cap \{\xi : r < |\xi - \varsigma| < r + \varrho\}$ . Then there is a constant  $C_1 > 0$  independent from  $l$  and  $r$  and a map  $w \in C_S^{0,1}(\bar{\Omega}, \mathbb{R}^m)$  such that*

- (i)  $w = v$  on  $\bar{\Omega} \setminus (C_{l+\frac{\lambda}{2}}^{r+\varrho}(\varsigma) \setminus \bar{C}_{l+\frac{\lambda}{2}}^r(\varsigma))$
- (ii)  $\|w - \bar{u}\|_{l+\frac{\lambda}{2}} \leq \bar{q}$ , for  $|\xi - \varsigma| = r + \frac{\varrho}{2}$ .
- (iii)  $J_{C_{l+\frac{\lambda}{2}}^{r+\varrho}(\varsigma) \setminus \bar{C}_{l+\frac{\lambda}{2}}^r(\varsigma)}(w) - J_{C_{l+\frac{\lambda}{2}}^{r+\varrho}(\varsigma) \setminus \bar{C}_{l+\frac{\lambda}{2}}^r(\varsigma)}(v) \leq C_1 \mathcal{H}^{n-1}(S)$ .

*Proof.* Set

$$(2.38) \quad \begin{aligned} q^v(\xi) &= \|v(\cdot, \xi) - \bar{u}(\cdot)\|_{l+\frac{\lambda}{2}}, \\ \nu^v(s, \xi) &= \frac{v(s, \xi) - \bar{u}(s)}{q^v(\xi)}, \quad \text{for } s \in (-l - \frac{\lambda}{2}, l + \frac{\lambda}{2}), \xi \in S. \end{aligned}$$

and, for  $s \in (-l - \frac{\lambda}{2}, l + \frac{\lambda}{2})$ ,  $\xi \in S$ , define

$$(2.39) \quad \begin{aligned} w(s, \xi) &= \bar{u}(s) + q^w(\xi) \nu^v(s, \xi), \\ q^w(\xi) &= (1 - |1 - 2 \frac{|\xi - \varsigma| - r}{\varrho}|) \bar{q} + |1 - 2 \frac{|\xi - \varsigma| - r}{\varrho}| q^v(\xi). \end{aligned}$$

From this definition it follows that  $w$  coincides with  $v = \bar{u} + q^v \nu^v$  if  $|\xi - \varsigma| = r$  or  $|\xi - \varsigma| = r + \varrho$  or  $q^v = \bar{q}$ . This shows that  $w$  coincides with  $v$  on the boundary of the set  $(-l - \frac{\lambda}{2}, l + \frac{\lambda}{2}) \times S$  and proves (i). From (2.39) also follows that  $q^w = \bar{q}$  for  $|\xi - \varsigma| = r + \frac{\varrho}{2}$  for  $\xi \in S$ . This and the definition of  $S$  imply (ii). To prove (iii) we note that

$$(2.40) \quad |w - \bar{u}| = |q^w \nu^v| \leq |q^v \nu^v| = |v - \bar{u}|, \quad \text{for } s \in (-l - \frac{\lambda}{2}, l + \frac{\lambda}{2}), \xi \in S.$$

which implies

$$(2.41) \quad |w - \bar{u}| \leq K e^{-ks}, \quad \text{for } s \in (0, l + \frac{\lambda}{2}), \xi \in S.$$

Therefore we have

$$(2.42) \quad \int_{-l-\frac{\lambda}{2}}^{l+\frac{\lambda}{2}} (W(w) - W(v)) \leq \int_{-l-\frac{\lambda}{2}}^{l+\frac{\lambda}{2}} W(w) \leq C, \quad \text{for } \xi \in S.$$

We can write

$$w = \frac{q^w}{q^v} (v - \bar{u}), \quad \text{for } s \in (0, l + \frac{\lambda}{2}), \xi \in S$$

therefore we have, using also (2.35)

$$(2.43) \quad \begin{aligned} w_s &= \frac{q^w}{q^v} (v_s - \bar{u}_s) \Rightarrow |w_s| \leq K e^{-k|s|}, \\ w_{\xi_j} &= (\frac{q^w}{q^v})_{\xi_j} (v - \bar{u}) + \frac{q^w}{q^v} v_{\xi_j}. \end{aligned}$$

From  $q_{\xi_j}^v = \langle \nu^v, v_{\xi_j} \rangle_{l+\frac{\lambda}{2}}$  and (2.39) it follows

$$(2.44) \quad \begin{aligned} (\frac{q^w}{q^v})_{\xi_j} &= |1 - 2 \frac{|\xi - \varsigma| - r}{\varrho}|_{\xi_j} (1 - \frac{\bar{q}}{q^v}) - (1 - |1 - 2 \frac{|\xi - \varsigma| - r}{\varrho}|) \frac{\bar{q}}{(q^v)^2} \langle \nu^v, v_{\xi_j} \rangle_{l+\frac{\lambda}{2}}, \\ \Rightarrow |(\frac{q^w}{q^v})_{\xi_j}| &\leq \frac{2}{\varrho} + \frac{1}{q^v} \|v_{\xi_j}\|_{l+\frac{\lambda}{2}}. \end{aligned}$$

where we have also used  $\frac{\bar{q}}{q^v} \leq 1$  for  $\xi \in S$ . From (2.44) and (2.44) it follows

$$|w_{\xi_j}| \leq \left(\frac{2}{\varrho} + \frac{\|v_{\xi_j}\|_{l+\frac{\lambda}{2}}}{\bar{q}}\right)|v - \bar{u}| + |v_{\xi_j}| \leq K e^{-k|s|}, \text{ for } s \in (-l - \frac{\lambda}{2}, l + \frac{\lambda}{2}), \xi \in S,$$

where we have also used (2.35). From this and (2.43) we conclude

$$(2.45) \quad \int_{-l-\frac{\lambda}{2}}^{l+\frac{\lambda}{2}} (|\nabla w|^2 - |\nabla v|^2) \leq \int_{-l-\frac{\lambda}{2}}^{l+\frac{\lambda}{2}} |\nabla w|^2 \leq C, \text{ for } \xi \in S.$$

This inequality together with (2.42) conclude the proof.  $\square$

**Lemma 2.5.** *Let  $w$  the map constructed in Lemma 2.4. Define  $w^{\bar{q}}$  by setting*

$$(2.46) \quad w^{\bar{q}} = \begin{cases} \bar{u} + \bar{q}\nu^v, & \text{for } (s, \xi) \in \mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{\varrho}{2}}(\varsigma), \xi \in A_{\bar{q}}, \\ w, & \text{for } (s, \xi) \in \mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{\varrho}{2}}(\varsigma), \xi \notin A_{\bar{q}}, \text{ and for } (s, \xi) \notin \mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{\varrho}{2}}(\varsigma). \end{cases}$$

Then  $w^{\bar{q}} \in C_S^{0,1}(\bar{\Omega}, \mathbb{R}^m)$  and

$$(2.47) \quad J_{\mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{\varrho}{2}}(\varsigma)}(w^{\bar{q}}) - J_{\mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{\varrho}{2}}(\varsigma)}(w) \leq 0.$$

*Proof.* We have  $w - \bar{u} = q^w \nu^w$  and  $q^w > \bar{q}$  on  $A_{\bar{q}}$ . Therefore, recalling the definition of  $\mathbf{e}_l$  and Lemma 2.1 we have

$$(2.48) \quad \begin{aligned} J_{\mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{\varrho}{2}}(\varsigma)}(w^{\bar{q}}) - J_{\mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{\varrho}{2}}(\varsigma)}(w) &= \int_{\tilde{A}_{\bar{q}}} (\mathbf{e}_{l+\frac{\lambda}{2}}(\bar{q}\nu^w) - \mathbf{e}_{l+\frac{\lambda}{2}}(q^w \nu^w)) d\xi \\ &\quad + \frac{1}{2} \sum_j \int_{\tilde{A}_{\bar{q}}} (\langle w_{\xi_j}^{\bar{q}}, w_{\xi_j}^{\bar{q}} \rangle_{l+\frac{\lambda}{2}} - \langle w_{\xi_j}, w_{\xi_j} \rangle_{l+\frac{\lambda}{2}}) d\xi \\ &\leq \frac{1}{2} \sum_j \int_{\tilde{A}_{\bar{q}}} (\langle w_{\xi_j}^{\bar{q}}, w_{\xi_j}^{\bar{q}} \rangle_{l+\frac{\lambda}{2}} - \langle w_{\xi_j}, w_{\xi_j} \rangle_{l+\frac{\lambda}{2}}) d\xi, \end{aligned}$$

To conclude the proof we note that for  $\xi \in \tilde{A}_{\bar{q}}$

$$(2.49) \quad \begin{aligned} w_{\xi_j}^{\bar{q}} &= \bar{q}\nu_{\xi_j}^v, \Rightarrow \langle w_{\xi_j}^{\bar{q}}, w_{\xi_j}^{\bar{q}} \rangle_{l+\frac{\lambda}{2}} = \bar{q}^2 \langle \nu_{\xi_j}^v, \nu_{\xi_j}^v \rangle_{l+\frac{\lambda}{2}}, \\ w_{\xi_j} &= q_{\xi_j}^w \nu + q^w \nu_{\xi_j}^v, \Rightarrow \langle w_{\xi_j}, w_{\xi_j} \rangle_{l+\frac{\lambda}{2}} = (q_{\xi_j}^w)^2 + (q^w)^2 \langle \nu_{\xi_j}^v, \nu_{\xi_j}^v \rangle_{l+\frac{\lambda}{2}} \end{aligned}$$

where we have also used that  $\langle \nu^v, \nu_{\xi_j}^v \rangle_{l+\frac{\lambda}{2}} = 0$ . Form (2.49) it follows

$$\langle w_{\xi_j}^{\bar{q}}, w_{\xi_j}^{\bar{q}} \rangle_{l+\frac{\lambda}{2}} - \langle w_{\xi_j}, w_{\xi_j} \rangle_{l+\frac{\lambda}{2}} = -(q_{\xi_j}^v)^2 + (\bar{q}^2 - (q^w)^2) \langle \nu_{\xi_j}^v, \nu_{\xi_j}^v \rangle_{l+\frac{\lambda}{2}} \leq 0,$$

for  $\xi \in \tilde{A}_{\bar{q}}$ . This and (2.48) prove (2.47).  $\square$

Next we show that we can associate to  $w^{\bar{q}}$  a map  $\omega$  which coincides with  $w^{\bar{q}}$  on  $\Omega \setminus \mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{\varrho}{2}}(\varsigma)$  and has less energy than  $w^{\bar{q}}$ . Moreover we derive a quantitative estimate

of the energy difference. We follow closely the argument in [18]. First we observe that, if we define  $q^* := q^{w^{\bar{q}}}$ , we can represent  $J_{\mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{g}{2}}(\varsigma)}(w^{\bar{q}})$  in the *polar* form

$$(2.50) \quad \begin{aligned} & J_{\mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{g}{2}}(\varsigma)}(w^{\bar{q}}) - J_{\mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{g}{2}}(\varsigma)}(\bar{u}) \\ &= \int_{B_{\varsigma, r+\frac{g}{2}} \cap \{q^* > 0\}} \frac{1}{2} (|\nabla q^*|^2 + q^{*2} \sum_j \langle \nu_{\xi_j}^w, \nu_{\xi_j}^w \rangle_{l+\frac{\lambda}{2}}) + \mathbf{e}_{l+\frac{\lambda}{2}}(q^* \nu^w). \end{aligned}$$

This follows from  $\nu^w = \nu^v$  and from  $\langle \nu^v, \nu_{\xi_j}^v \rangle_{l+\frac{\lambda}{2}} = 0$  that implies

$$\sum_j \langle w_{\xi_j}^{\bar{q}}, w_{\xi_j}^{\bar{q}} \rangle_{l+\frac{\lambda}{2}} = |\nabla q^*|^2 + q^{*2} \sum_j \langle \nu_{\xi_j}^w, \nu_{\xi_j}^w \rangle_{l+\frac{\lambda}{2}}$$

and from the definition of  $\mathbf{e}_l$  in Lemma 2.1. We remark that the definition of  $q^*$  and  $w^{\bar{q}}$  imply

$$(2.51) \quad \begin{aligned} q^* &\leq \bar{q}, \text{ on } B_{\varsigma, r+\frac{g}{2}}, \\ q^* &= \bar{q}, \text{ on } A_{\bar{q}} \cap B_{\varsigma, r+\frac{g}{2}}. \end{aligned}$$

**Lemma 2.6.** *Let  $\varphi : B_{\varsigma, r+\frac{g}{2}} \rightarrow \mathbb{R}$  the solution of*

$$(2.52) \quad \begin{cases} \Delta \varphi = c^2 \varphi, & \text{in } B_{\varsigma, r+\frac{g}{2}} \\ \varphi = \bar{q}, & \text{on } \partial B_{\varsigma, r+\frac{g}{2}}. \end{cases}$$

*Then there is a map  $\omega \in C_S^{0,1}(\bar{\Omega}, \mathbb{R}^m)$  with the following properties*

$$(2.53) \quad \begin{cases} \omega = w^{\bar{q}}, & \text{on } \Omega \setminus \mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{g}{2}}(\varsigma), \\ \omega = q^\omega \nu^w + \bar{u}, & \text{on } \mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{g}{2}}(\varsigma), \\ q^\omega \leq \varphi \leq \bar{q}, & \text{on } \mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{g}{2}}(\varsigma). \end{cases}$$

*Moreover*

$$(2.54) \quad \begin{aligned} & J_{\mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{g}{2}}(\varsigma)}(w^{\bar{q}}) - J_{\mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{g}{2}}(\varsigma)}(\omega) \\ & \geq \int_{B_{\varsigma, r+\frac{g}{2}} \cap \{q^* > \varphi\}} (\mathbf{e}_{l+\frac{\lambda}{2}}(q^* \nu^w) - \mathbf{e}_{l+\frac{\lambda}{2}}(\varphi \nu^w) - D_q \mathbf{e}_{l+\frac{\lambda}{2}}(\varphi \nu^w)(q^* - \varphi)) d\xi. \end{aligned}$$

*Proof.* Let  $b > 0$ ,  $b \leq \min_{\xi \in B_{\varsigma, r+\frac{g}{2}}} \varphi$  be fixed and let  $A_b \subset B_{\varsigma, r+\frac{g}{2}}$  the set  $A_b := \{\xi \in B_{\varsigma, r+\frac{g}{2}} : q^* > b\}$ .  $A_b$  is an open set since  $w^{\bar{q}} = \bar{u} + q^* \nu^w$  is continuous by construction. Let

$$(2.55) \quad \mathcal{J}_{A_b}(p) = \int_{A_b} \left( \frac{1}{2} |\nabla p|^2 + \mathbf{e}_{l+\frac{\lambda}{2}}(|p| \nu^w) \right) d\xi,$$

Since  $A_b$  is open and  $q^* \in L^\infty(A_b, \mathbb{R})$  there exists a minimizer  $p^* \in q^* + W_0^{1,2}(A_b, \mathbb{R})$  of the problem

$$(2.56) \quad \mathcal{J}_{A_b}(p^*) = \min_{q^* + W_0^{1,2}(A_b, \mathbb{R})} \mathcal{J}_{A_b}.$$

We also have

$$(2.57) \quad 0 \leq p^* \leq \bar{q}.$$

This follows from (2.7) that implies  $\mathcal{J}_{A_b}(\frac{p^*+|p^*|}{2}) \leq \mathcal{J}_{A_b}(p^*)$  and therefore  $p^* \geq 0$ . The other inequality is a consequence of  $\mathcal{J}_{A_b}(\min\{p^*, \bar{q}\}) \leq \mathcal{J}_{A_b}(p^*)$  which follows from  $\int_{A_b} |\nabla(\min\{p^*, \bar{q}\})|^2 \leq \int_{A_b} |\nabla p^*|^2$  and from (2.7). Since the map  $q \rightarrow \mathbf{e}_{l+\frac{\lambda}{2}}(|q|\nu^w)$  is a  $C^1$  map, we can write the variational equation

$$(2.58) \quad \int_{A_b} ((\nabla p^*, \nabla \gamma) + D_q \mathbf{e}_{l+\frac{\lambda}{2}}(p^* \nu^w) \gamma) d\xi = 0,$$

for all  $\gamma \in W_0^{1,2}(A_b, \mathbb{R}) \cap L^\infty(A_b)$ . In particular, if we define  $A_b^* := \{x \in A_b : p^* > \varphi\}$ , we have

$$(2.59) \quad \int_{A_b^*} ((\nabla p^*, \nabla \gamma) + D_q \mathbf{e}_{l+\frac{\lambda}{2}}(p^* \nu^w) \gamma) d\xi = 0,$$

for all  $\gamma \in W_0^{1,2}(A_b, \mathbb{R}) \cap L^\infty(A_b)$  that vanish on  $A_b \setminus A_b^*$ . If we take  $\gamma = (p^* - \varphi)^+$  in (2.59) and use (2.7)<sub>2</sub> which implies  $D_q \mathbf{e}_{l+\frac{\lambda}{2}}(p^* \nu^w) \geq c^2 p^*$  we get

$$(2.60) \quad \int_{A_b^*} ((\nabla p^*, \nabla(p^* - \varphi)) + c^2 p^*(p^* - \varphi)) d\xi \leq 0,$$

This inequality and

$$(2.61) \quad \int_{A_b^*} ((\nabla \varphi, \nabla(p^* - \varphi)) + c^2 \varphi(p^* - \varphi)) dx = 0,$$

that follows from (2.52) imply

$$(2.62) \quad \int_{A_b^*} (|\nabla(p^* - \varphi)|^2 + c^2(p^* - \varphi)^2) d\xi \leq 0.$$

That is  $\mathcal{H}^n(A_b^*) = 0$  which together with  $p^* \leq \varphi$  on  $A_b \setminus A_b^*$  shows that

$$(2.63) \quad p^* \leq \varphi, \text{ for } \xi \in A_b.$$

Let  $\omega$  be the map defined by setting

$$(2.64) \quad \omega = \begin{cases} w^{\bar{q}}, & \text{for } (s, \xi) \in \Omega \setminus (-l - \frac{\lambda}{2}, l + \frac{\lambda}{2}) \times A_b, \\ \bar{u} + q^\omega \nu^w = \bar{u} + \min\{p^*, q^*\} \nu^w, & \text{for } \xi \in A_b. \end{cases}$$

Note that this definition, the definition of  $A_b$  and (2.63) imply

$$(2.65) \quad q^\omega \leq \varphi, \text{ for } \xi \in B_{s, r+\frac{\varrho}{2}}.$$

From (2.64) we have

$$\begin{aligned}
(2.66) \quad & J_{\mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{\varrho}{2}}(\varsigma)}(w^{\bar{q}}) - J_{\mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{\varrho}{2}}(\varsigma)}(\omega) \\
& \geq \int_{A_b \cap \{p^* < q^*\}} \left( \frac{1}{2} (|\nabla q^*|^2 - |\nabla p^*|^2 + ((q^*)^2 - (p^*)^2)) \sum_{j=1}^n \langle \nu_{\xi_j}^w, \nu_{\xi_j}^w \rangle_{l+\frac{\lambda}{2}} \right. \\
& \quad \left. + \mathbf{e}_{l+\frac{\lambda}{2}}(q^* \nu^w) - \mathbf{e}_{l+\frac{\lambda}{2}}(p^* \nu^w) \right) d\xi \\
& \geq \int_{A_b \cap \{p^* < q^*\}} \left( \frac{1}{2} (|\nabla q^*|^2 - |\nabla p^*|^2 + \mathbf{e}_{l+\frac{\lambda}{2}}(q^* \nu^w) - \mathbf{e}_{l+\frac{\lambda}{2}}(p^* \nu^w)) \right) d\xi \\
& \geq \int_{A_b \cap \{p^* < q^*\}} \left( \frac{1}{2} |\nabla q^* - \nabla p^*|^2 \right. \\
& \quad \left. + \mathbf{e}_{l+\frac{\lambda}{2}}(q^* \nu^w) - \mathbf{e}_{l+\frac{\lambda}{2}}(p^* \nu^w) \right) d\xi - D_q \mathbf{e}_{l+\frac{\lambda}{2}}(p^* \nu^w)(q^* - p^*) d\xi \geq 0.
\end{aligned}$$

where we have used

$$(2.67) \quad \frac{1}{2} (|\nabla q^*|^2 - |\nabla p^*|^2) = \frac{1}{2} |\nabla q^* - \nabla p^*|^2 + (\nabla p^*, \nabla(q^* - p^*)),$$

and

$$\int_{A_b \cap \{p^* < q^*\}} (\nabla p^*, \nabla(q^* - p^*)) = - \int_{A_b \cap \{p^* < q^*\}} D_q \mathbf{e}_{l+\frac{\lambda}{2}}(p^* \nu^w)(q^* - p^*) d\xi,$$

which follows from (2.58) with  $\gamma = (q^* - p^*)^+$ . From (2.73) and (2.63) we have

$$(2.68) \quad \mathbf{e}_{l+\frac{\lambda}{2}}(q^* \nu^w) - \tilde{\mathbf{e}}_{l+\frac{\lambda}{2}}(p^*, q^*, \nu^w) \geq \mathbf{e}_{l+\frac{\lambda}{2}}(q^* \nu^w) - \tilde{\mathbf{e}}_{l+\frac{\lambda}{2}}(\varphi, q^*, \nu^w).$$

From this and (2.65) which implies

$$(2.69) \quad B_{\varsigma, r+\frac{\varrho}{2}} \cap \{\phi < q^*\} = A_b \cap \{\phi < q^*\} \subset A_b \cap \{p^* < q^*\},$$

we have

$$\begin{aligned}
(2.70) \quad & \int_{A_b \cap \{p^* < q^*\}} \mathbf{e}_{l+\frac{\lambda}{2}}(q^* \nu^w) - \mathbf{e}_{l+\frac{\lambda}{2}}(p^* \nu^w) - D_q \mathbf{e}_{l+\frac{\lambda}{2}}(p^* \nu^w)(q^* - p^*) d\xi \\
& \geq \int_{B_{\varsigma, r+\frac{\varrho}{2}} \cap \{\varphi < q^*\}} \mathbf{e}_{l+\frac{\lambda}{2}}(q^* \nu^w) - \mathbf{e}_{l+\frac{\lambda}{2}}(p^* \nu^w) - D_q \mathbf{e}_{l+\frac{\lambda}{2}}(p^* \nu^w)(q^* - p^*) d\xi \\
& \geq \int_{B_{\varsigma, r+\frac{\varrho}{2}} \cap \{\varphi < q^*\}} \mathbf{e}_{l+\frac{\lambda}{2}}(q^* \nu^w) - \mathbf{e}_{l+\frac{\lambda}{2}}(\varphi \nu^w) - D_q \mathbf{e}_{l+\frac{\lambda}{2}}(\varphi \nu^w)(q^* - \varphi) d\xi.
\end{aligned}$$

The inequality (2.54) follows from this and (2.66).  $\square$

**Corollary 2.7.** *Let  $w^{\bar{q}}$  as before and let  $\omega \in C_S^{0,1}(\bar{\Omega}, \mathbb{R}^m)$  the map constructed in Lemma 2.6. Then there is a number  $c_1 > 0$  independent from  $l, r, \lambda$  and  $\varrho$  such that*

$$(2.71) \quad J_{\mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{\varrho}{2}}(\varsigma)}(w^{\bar{q}}) - J_{\mathcal{C}_{l+\frac{\lambda}{2}}^{r+\frac{\varrho}{2}}(\varsigma)}(\omega) \geq c_1 \mathcal{H}^{n-1}(A_{\bar{q}} \cap B_{\varsigma, r}).$$

*Proof.* Set  $R = r + \frac{\varrho}{2}$ , then we have  $\varphi(\xi) = \bar{q}\phi(|\xi - \varsigma|, R)$  with  $\phi(\cdot, R) : [0, R] \rightarrow \mathbb{R}$  a positive function which is strictly increasing in  $(0, R]$ . Moreover we have  $\phi(R, R) = 1$  and

$$(2.72) \quad R_1 < R_2, t \in (0, R_1) \Rightarrow \phi(R_1 - t, R_1) > \phi(R_2 - t, R_2).$$

Note that  $\xi \in B_{\varsigma, r}$  implies  $\varphi(\xi) \leq \bar{q}\phi(r, r + \frac{\varrho}{2})$ . Therefore for  $\xi \in B_{\varsigma, r} \cap A_{\bar{q}}$  we have

$$(2.73) \quad \begin{aligned} & \mathbf{e}_{l+\frac{\lambda}{2}}(\bar{q}\nu^w) - \mathbf{e}_{l+\frac{\lambda}{2}}(\varphi\nu^w) - D_q \mathbf{e}_{l+\frac{\lambda}{2}}(\varphi\nu^w)(\bar{q} - \varphi) \\ &= \int_{\varphi}^{\bar{q}} (D_q \mathbf{e}_{l+\frac{\lambda}{2}}(s\nu^w) - D_q \mathbf{e}_{l+\frac{\lambda}{2}}(\varphi\nu^w)) ds \\ &\geq c^2 \int_{\varphi}^{\bar{q}} (s - \varphi) ds = \frac{1}{2} c^2 (\bar{q} - \varphi)^2 \geq \frac{1}{2} c^2 \bar{q}^2 (1 - \phi(r, r + \frac{\varrho}{2}))^2, \end{aligned}$$

where we have also used (2.7)<sub>1</sub>. The corollary follows from this inequality, from (2.54) and from the fact that, by (2.72), the last expression in (2.73) is increasing with  $r$ . Therefore, for  $r \geq r_0$ , for some  $r_0 > 0$ , we can assume

$$(2.74) \quad c_1 = \frac{1}{2} c^2 \bar{q}^2 (1 - \phi(r_0, r_0 + \frac{\varrho}{2}))^2.$$

□

### 2.3 Conclusion of the proof of Theorem 1.2

Let  $u$  as in Theorem 1.2 and  $l_0, q^\circ$  as in Lemma 2.1 and assume that  $\varsigma$  is such that

$$(2.75) \quad \|u(\cdot, \varsigma) - \bar{u}\|_l \geq q^\circ,$$

for some  $l \geq l_0$ . Then  $u \in C_S^{0,1}(\bar{\Omega}, \mathbb{R}^m)$  implies that, there is  $r_0 > 0$  independent from  $l \geq l_0$  such that,

$$(2.76) \quad \|u(\cdot, \xi) - \bar{u}\|_l \geq \bar{q}, \text{ for } |\xi - \varsigma| \leq r_0.$$

Let  $j_0 \geq 0$ , be minimum value of  $j$  that violated the inequality

$$(2.77) \quad c_1 \frac{r_0^{n-1}}{2} (1 + \frac{c_1}{C_1})^j \leq C_1 ((r_0 + (j+1)\varrho)^{n-1} - (r_0 + j\varrho)^{n-1}),$$

where  $c_1$  and  $C_2$  are the constants in Corollary 2.7 and Lemma 2.4. Let  $l^\circ \geq l_0$  be fixed so that

$$(2.78) \quad C_0(r_0 + j_0\varrho)^{n-1} e^{-kl^\circ} \leq c_1 \theta_{n-1} \frac{r_0^{n-1}}{2},$$

where  $C_0$  is defined in Lemma 2.3 and  $\theta_n$  is the measure of the unit ball in  $\mathbb{R}^n$ ,

**Proposition 2.8.** *Let  $\lambda, \varrho, \bar{q} \in (0, q^\circ)$  and  $l^\circ \geq l_0$  fixed as before and let  $r^\circ = r_0 + j_0\varrho$  where  $j_0 \geq 0$  is the minimum value of  $j$  that violates (2.77). Assume  $l \geq l^\circ$  and assume that  $\mathcal{C}_{l+\lambda}^{r^\circ+2\varrho}(\varsigma) \subset \Omega$  satisfies*

$$(2.79) \quad d(\mathcal{C}_{l+\lambda}^{r^\circ+2\varrho}(\varsigma), \partial\Omega) \geq l + \lambda.$$

Then

$$(2.80) \quad q^u(\varsigma) = \|u(\cdot, \varsigma) - \bar{u}\|_{l+\frac{\lambda}{2}} < q^\circ.$$



*Proof.* Suppose instead that

$$(2.81) \quad \|u(\cdot, \varsigma) - \bar{u}\|_{l+\frac{\lambda}{2}} \geq q^\circ,$$

and set

$$(2.82) \quad \sigma_0 := \theta_{n-1} \frac{r_0^{n-1}}{2}.$$

Then  $l^\circ \geq l_0$  and (2.76) imply

$$(2.83) \quad \mathcal{H}^{n-1}(A_{\bar{q}} \cap B_{\varsigma, r_0}) \geq 2\sigma_0.$$

For each  $0 \leq j \leq j_0$  let  $r_j := r_0 + j\varrho$  and let  $v_j, w_j, w_j^{\bar{q}}$  and  $\omega_j$  the maps  $v, w, w^{\bar{q}}$  and  $\omega$  defined in Lemma 2.3, Lemma 2.4, Lemma 2.5 and Lemma 2.6 with  $l \geq l^\circ$  and  $r = r_j$ . Then from these Lemmas and Corollary 2.7 we have

$$(2.84) \quad \begin{aligned} J(u)_{\mathcal{C}_{l+\lambda}^{r_j^\circ+2\varrho}(\varsigma)} - J(v_j)_{\mathcal{C}_{l+\lambda}^{r_j^\circ+2\varrho}(\varsigma)} &\geq -C_0 r_j^{n-1} e^{-kl^\circ}, \\ J(v_j)_{\mathcal{C}_{l+\lambda}^{r_j^\circ+2\varrho}(\varsigma)} - J(w_j)_{\mathcal{C}_{l+\lambda}^{r_j^\circ+2\varrho}(\varsigma)} &\geq -C_1 \mathcal{H}^{n-1}(A_{\bar{q}} \cap (\overline{B}_{\varsigma, r_{j+1}} \setminus B_{\varsigma, r_j})), \\ J(w_j)_{\mathcal{C}_{l+\lambda}^{r_j^\circ+2\varrho}(\varsigma)} - J(w_j^{\bar{q}})_{\mathcal{C}_{l+\lambda}^{r_j^\circ+2\varrho}(\varsigma)} &\geq 0, \\ J(w_j^{\bar{q}})_{\mathcal{C}_{l+\lambda}^{r_j^\circ+2\varrho}(\varsigma)} - J(\omega_j)_{\mathcal{C}_{l+\lambda}^{r_j^\circ+2\varrho}(\varsigma)} &\geq c_1 \mathcal{H}^{n-1}(A_{\bar{q}} \cap \overline{B}_{\varsigma, r_j}). \end{aligned}$$

From this and the minimality of  $u$  it follows

$$(2.85) \quad 0 \geq -C_0 r_j^{n-1} e^{-kl^\circ} - C_1 \mathcal{H}^{n-1}(A_{\bar{q}} \cap (\overline{B}_{\varsigma, r_{j+1}} \setminus B_{\varsigma, r_j})) + c_1 \mathcal{H}^{n-1}(A_{\bar{q}} \cap \overline{B}_{\varsigma, r_j}).$$

Define

$$(2.86) \quad \sigma_j := \mathcal{H}^{n-1}(A_{\bar{q}} \cap B_{\varsigma, r_j}) - \sigma_0, \text{ for } j \geq 1.$$

If  $j_0 = 0$  the inequality (2.85), using also (2.78), implies

$$(2.87) \quad 0 \geq -c_1 \sigma_0 - C_1 \sigma_1 + 2C_1 \sigma_0 + 2c_1 \sigma_0 \geq c_1 \sigma_0 - C_1(\sigma_1 - \sigma_0).$$

If  $j_0 > 0$  in a similar way we get

$$(2.88) \quad 0 \geq -c_1 \sigma_0 - C_1(\sigma_{j-1} - \sigma_j) + c_1(\sigma_j + \sigma_0) = c_1 \sigma_j - C_1(\sigma_{j+1} - \sigma_j).$$

From (2.87) and (2.88) it follows

$$(2.89) \quad \sigma_j \geq (1 + \frac{c_1}{C_1})^j \sigma_0,$$

and therefore, using also (2.82)

$$(2.90) \quad c_1(1 + \frac{c_1}{C_1})^j \theta_{n-1} \frac{r_0^{n-1}}{2} \leq C_1(\sigma_{j+1} - \sigma_j) \leq C_1 \theta_{n-1} (r_{j+1}^{n-1} - r_j^{n-1}).$$

This inequality is equivalent to (2.77). It follows that, on the basis of the definition of  $j_0$ , putting  $j = j_0$  in (2.90) leads to a contradiction with the minimality of  $u$ .  $\square$

## 2.4 The exponential estimate

**Lemma 2.9.** Assume  $r > r^\circ + 2\varrho$  and  $l > l^\circ + \lambda$  and assume that  $\mathcal{C}_l^r(\varsigma_0) \subset \Omega$  satisfies

$$(2.91) \quad d(\mathcal{C}_l^r(\varsigma_0), \partial\Omega) \geq l.$$

Then there are constants  $K_1$  and  $k_1 > 0$  independent of  $r > r^\circ + 2\varrho$  and  $l > l^\circ + \lambda$  such that

$$(2.92) \quad \|u(\cdot, \varsigma_0) - \bar{u}\|_l^{\frac{1}{2}} \leq K_1 e^{-k_1 r}.$$

*Proof.* From  $r > r^\circ + 2\varrho$  it follows that  $|\varsigma - \varsigma_0| \leq r - (r^\circ + 2\varrho)$  implies

$$(2.93) \quad d(\mathcal{C}_l^{r^\circ+2\varrho}(\varsigma), \partial\Omega) \geq l.$$

Therefore we can invoke Proposition 2.8 to conclude that

$$(2.94) \quad \|u(\cdot, \varsigma) - \bar{u}\| \leq \bar{q}, \text{ for } |\varsigma - \varsigma_0| \leq r - (r^\circ + 2\varrho).$$

Let  $\varphi : B_{\varsigma_0, r-(r^\circ+2\varrho)} \rightarrow \mathbb{R}$  the solution of

$$(2.95) \quad \begin{cases} \Delta\varphi = c^2\varphi, & \text{in } B_{\varsigma_0, r-(r^\circ+2\varrho)} \\ \varphi = \bar{q}, & \text{on } \partial B_{\varsigma_0, r-(r^\circ+2\varrho)}. \end{cases}$$

Then we have

$$(2.96) \quad \|u(\cdot, \varsigma) - \bar{u}\| \leq \varphi(\varsigma), \text{ for } \varsigma \in B_{\varsigma_0, r-(r^\circ+2\varrho)}.$$

This follows by the same argument leading to (2.65) in the proof of Lemma 2.6. Indeed, if (2.96) does not hold, then by proceeding as in the proof of Lemma 2.6 we can construct a competing map  $\omega$  that satisfies (2.96) and has less energy than  $u$  contradicting its minimality property. In particular (2.96) implies

$$(2.97) \quad \|u(\cdot, \varsigma_0) - \bar{u}\| \leq \varphi(\varsigma_0).$$

On the other hand it can be shown, see Lemma 2.4 in [19], that there is a constant  $h_0 > 0$  such that

$$\phi(0, r) \leq e^{-h_0 r}; \text{ for } r \geq r_0$$

From this and (2.97) we get

$$(2.98) \quad \varphi(\varsigma_0) = \bar{q}\phi(0, r - (r^\circ + 2\varrho)) \leq \bar{q}e^{h_0(r^\circ+2\varrho)}e^{-h_0 r} = K_1 e^{-k_1 r}.$$

This concludes the proof with  $K_1 = \bar{q}e^{h_0(r^\circ+2\varrho)}$  and  $k_1 = h_0$ .  $\square$

We are now in the position of proving the exponential estimate (i) in Theorem 1.2. We distinguish two cases:

**Case 1**  $x = (s, \xi) \in \Omega$  satisfies  $s > \frac{1}{2}d(x, \partial\Omega)$ . In this case, taking also into account that  $\Omega$  satisfies (i), we have

$$(2.99) \quad d(x, \partial\Omega^+) \geq \frac{1}{2}d(x, \partial\Omega).$$

From this and Theorem 1.1 it follows

$$(2.100) \quad |u(s, \xi) - \bar{u}(s)| \leq |u(s, \xi) - a| + |\bar{u}(s) - a| \\ \leq K_0 e^{-k_0 d(x, \partial\Omega^+)} + \bar{K} e^{-\bar{k}s} \leq (K_0 + \bar{K}) e^{-\frac{1}{2} \min\{k_0, \bar{k}\} d(x, \partial\Omega)},$$

where we have also used

$$(2.101) \quad |\bar{u}(s) - a| \leq \bar{K} e^{-\bar{k}s}.$$

**Case 2**  $x = (s, \xi) \in \Omega$  satisfies  $0 \leq s \leq \frac{1}{2}d(x, \partial\Omega)$ . In this case, elementary geometric considerations and the assumption (i) on  $\Omega$  imply the existence of  $\alpha \in (0, 1)$  ( $\alpha = \frac{1}{4}$  will do) such that

$$(2.102) \quad \mathcal{C}_{s+\alpha d(x)}^{\alpha d(x)}(\xi) \subset \Omega \quad \text{and} \\ d(\mathcal{C}_{s+\alpha d(x)}^{\alpha d(x)}(\xi), \partial\Omega) \geq s + \alpha d(x),$$

where we have set  $d(x) := d(x, \partial\Omega)$ . From (2.102) and Lemma 2.9 it follows

$$(2.103) \quad \|u(\cdot, \xi) - \bar{u}\|_l \leq K_1 e^{-k_1 \alpha d(x)}, \quad \text{for } d(x) > r^\circ + 2\rho.$$

This and Lemma 2.2 imply, recalling  $d(x) = d(x, \partial\Omega)$ ,

$$(2.104) \quad |u(s, \xi) - \bar{u}(s)| \leq K_1^{\frac{2}{3}} e^{-\frac{2}{3} k_1 \alpha d(x, \partial\Omega)}.$$

The exponential estimate follows from (2.104) and (2.104).

## 2.5 The proof of Theorems 1.3 and 1.4

If  $\Omega = \mathbb{R}^n$  the proof of Theorem 1.2 simplifies since we can avoid the technicalities needed in the case that  $\Omega$  is bounded in the  $s = x_1$  direction and assume  $l = +\infty$ . The possibility of working with  $l = +\infty$  is based on the following lemma

**Lemma 2.10.** *Let  $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$  the symmetric minimizer in Theorem 1.1. Given a smooth open set  $O \subset \mathbb{R}^{n-1}$  let  $\mathbb{R} \times O$  the cylinder  $\mathbb{R} \times O = \{(s, \xi) : s \in \mathbb{R}, \xi \in O\}$ . Then*

$$(2.105) \quad J_{\mathbb{R} \times O}(u) = \min_{v \in u + W_{0S}^{1,2}(\mathbb{R} \times O; \mathbb{R}^m)} J_{\mathbb{R} \times O}(v),$$

where  $W_{0S}^{1,2}(\mathbb{R} \times O; \mathbb{R}^m)$  is the subset of  $W_S^{1,2}(\mathbb{R} \times O; \mathbb{R}^m)$  of the maps that satisfy  $v = 0$  on  $\partial\mathbb{R} \times O$ .

*Proof.* Assume there are  $\eta > 0$  and  $v \in W_{0S}^{1,2}(\mathbb{R} \times O; \mathbb{R}^m)$  such that

$$(2.106) \quad J_{\mathbb{R} \times O}(u) - J_{\mathbb{R} \times O}(v) \geq \eta.$$

For each  $l > 0$  define  $\tilde{v} \in W_{0S}^{1,2}(\mathbb{R} \times O; \mathbb{R}^m)$  by

$$\tilde{v} = \begin{cases} v, & \text{for } s \in [0, l], \xi \in O, \\ (1 + l - s)v + (s - l)u, & s \in [l, l + 1], \xi \in O, \\ u, & \text{for } s \in [l, +\infty), \xi \in O. \end{cases}$$

The minimality of  $u$  implies

$$(2.107) \quad 0 \geq J_{[-l-1, l+1] \times O}(u) - J_{[-l-1, l+1] \times O}(\tilde{v}) = J_{[-l-1, l+1] \times O}(u) - J_{[-l, l] \times O}(v) + O(e^{-kl}),$$

where we have also used the fact that both  $u$  and  $v$  belong to  $W_S^{1,2}(\mathbb{R} \times O; \mathbb{R}^m)$ . Taking the limit for  $l \rightarrow +\infty$  in (2.107) yields

$$0 \geq J_{\mathbb{R} \times O}(u) - J_{\mathbb{R} \times O}(v)$$

in contradiction with (2.106).  $\square$

Once we know that  $u$  satisfies (2.105) the same arguments leading to Proposition 2.8 imply the existence of  $r^\circ > 0$  such that

$$(2.108) \quad \mathbb{R} \times B_{r^\circ}(\xi) \subset \mathbb{R}^n \Rightarrow \|u(\cdot, \xi) - \bar{u}\|_\infty < q^\circ,$$

where  $B_{r^\circ}(\xi) \subset \mathbb{R}^{n-1}$  is the ball of center  $\xi$  and radius  $r^\circ$ . Since the condition  $\mathbb{R} \times B_{r^\circ}(\xi) \subset \mathbb{R}^n$  is trivially satisfied for each  $\xi \in \mathbb{R}^{n-1}$  we have

$$\|u(\cdot, \xi) - \bar{u}\|_\infty < q^\circ, \quad \text{for every } \xi \in \mathbb{R}^{n-1}.$$

To conclude the proof we observe that everything has been said concerning  $q^\circ$  can be repeated verbatim for each  $q \in (0, q^\circ)$ . It follows that for each  $q \in (0, q^\circ]$  there is a  $r(q) > 0$  such that (2.108) holds with  $q$  in place of  $q^\circ$  and  $r(q)$  in place of  $r^\circ$ . Therefore we have

$$\|u(\cdot, \xi) - \bar{u}\|_\infty < q, \quad \text{for every } \xi \in \mathbb{R}^{n-1}.$$

Since this holds for each  $q \in (0, q^\circ]$  we conclude

$$u(\cdot, \xi) = \bar{u}, \quad \text{for every } \xi \in \mathbb{R}^{n-1}$$

which complete the proof of Theorem 1.3.

To prove Theorem 1.4 we note that, if  $\Omega = \{x \in \mathbb{R}^n : x_n > 0\}$ , then arguing as in the proof of Theorem 1.3 above, we get that, given  $q > 0$  there exists  $l_q > 0$  such that

$$\xi_n > l_q, \quad \Rightarrow \quad \|u(\cdot, \xi) - \bar{u}\|_{L^\infty} < q.$$

From this, the boundary condition

$$\xi_n = 0, \quad \Rightarrow \quad \|u(\cdot, \xi) - \bar{u}\|_{L^\infty} = 0,$$

and the reasoning in the proof of Lemma 2.5 it follows

$$\|u(\cdot, \xi) - \bar{u}\|_{L^\infty} < q, \quad \text{for each } \xi_n \geq 0, q > 0.$$

The proof of Theorem 1.4 is complete.

### 3 The proof of Theorem 1.5

From an abstract point of view the proof of Theorem 1.5 is essentially the same as the proof of Theorem 1.3 after quantities like  $q^u$  and  $\nu^u$  are reinterpreted and properly redefined in the context of maps equivariant with respect to the group  $G$  of the equilateral triangle. We divide the proof in steps pointing out the correspondence with the corresponding steps in the proof of Theorem 1.3. We write  $x \in \mathbb{R}^n$  in the form  $x = (s, \xi)$  with  $s = (s_1, s_2) \in \mathbb{R}^2$  and  $\xi = (x_2, \dots, x_n) \in \mathbb{R}^{n-2}$ .

### Step 1

From assumption (1.23) in Theorem 1.5 and equivariance it follows

$$(3.1) \quad \begin{aligned} |u(x) - a| &\geq \delta, \quad |u(x) - g_- a| > \delta, \quad \text{for } x \in g_+ D, \quad d(x, \partial g_+ D) \geq d_0, \\ |u(x) - a| &\geq \delta, \quad |u(x) - g_+ a| > \delta, \quad \text{for } x \in g_- D, \quad d(x, \partial g_- D) \geq d_0. \end{aligned}$$

From this and assumptions  $\mathbf{H}'_3$  and  $\mathbf{H}'_4$  it follows that we can apply Theorem 1.2 with  $\Omega = \mathbb{R}^n \setminus \overline{D}$  and  $a_\pm = g_\pm a$  to conclude that there exist  $k, K > 0$  such that

$$(3.2) \quad |u(s_1, s_2, \xi) - \bar{u}(s_2)| \leq K e^{-kd(x, \partial(\mathbb{R}^n \setminus \overline{D}))}, \quad x \in \mathbb{R}^n \setminus \overline{D}.$$

In exactly the same way we establish that

$$(3.3) \quad |\tilde{u}(s_1, s_2) - \bar{u}(s_2)| \leq K e^{-kd(s, \partial(\mathbb{R}^2 \setminus \overline{D_2}))}, \quad s \in \mathbb{R}^2 \setminus \overline{D_2},$$

where  $D_2 \subset \mathbb{R}^2 = \{s : |s_2| < \sqrt{3}s_1, \quad s_1 > 0\}$ . From (3.2), (3.3) and equivariance it follows

$$(3.4) \quad |u(s, \xi) - \tilde{u}(s)| \leq K e^{-k|s|}, \quad \text{for } s \in \mathbb{R}^2, \quad \xi \in \mathbb{R}^{n-2}.$$

### Step 2

Let  $C_G^{0,1}(\mathbb{R}^n; \mathbb{R}^m)$  the set of lipshitz maps  $v : \mathbb{R}^n \rightarrow \mathbb{R}^m$  which are equivariant under  $G$  and satisfy

$$(3.5) \quad \begin{aligned} |v(s, \xi) - \tilde{u}(s)| &\leq K e^{-k|s|}, \\ |\nabla_s v(s, \xi) - \nabla_s \tilde{u}(s)| &\leq K e^{-k|s|}, \quad \text{for } s \in \mathbb{R}^2, \quad \xi \in \mathbb{R}^{n-2}, \\ |\nabla_\xi v(s, \xi)| &\leq K e^{-k|s|}, \end{aligned}$$

We remark that from (3.4) we have  $u \in C_G^{0,1}(\mathbb{R}^n; \mathbb{R}^m)$  for the minimizer  $u$  in Theorem 1.5. If  $O \subset \mathbb{R}^{n-2}$  is an open bounded set with a lipshitz boundary we let  $C_G^{0,1}(\mathbb{R}^2 \times O; \mathbb{R}^m)$  the set of equivariant maps that satisfy (3.5) for  $\xi \in O$ . We denote  $C_{0,G}^{0,1}(\mathbb{R}^2 \times O; \mathbb{R}^m)$  the subset of  $C_G^{0,1}(\mathbb{R}^2 \times O; \mathbb{R}^m)$  of the maps the vanish on the boundary of  $\mathbb{R}^2 \times O$ . The spaces  $W_G^{1,2}(\mathbb{R}^2 \times O; \mathbb{R}^m)$  and  $W_{0,G}^{1,2}(\mathbb{R}^2 \times O; \mathbb{R}^m)$  are defined in the obvious way. The exponential estimates in the definition of these function spaces and the same argument in the proof of Lemma 2.10 imply

**Lemma 3.1.** *Let  $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$  the  $G$ -equivariant minimizer in Theorem 1.5. Given an open bounded lipshitz set  $O \subset \mathbb{R}^{n-2}$  we have*

$$(3.6) \quad J_{\mathbb{R}^2 \times O}(u) = \min_{v \in u + W_{0,G}^{1,2}(\mathbb{R}^2 \times O; \mathbb{R}^m)} J_{\mathbb{R}^2 \times O}(v),$$

### Step 3

In analogy with the definition of  $\mathbf{e}(v)$  in Lemma 2.1, for  $v \in W_G^{1,2}(\mathbb{R}^n; \mathbb{R}^m)$ , we define the *effective potential*  $\mathbf{E}(v)$  for the case at hand. We set

$$(3.7) \quad \mathbf{E}(v) = \frac{1}{2}(\langle \nabla_s \tilde{u} + \nabla_s v, \nabla_s \tilde{u} + \nabla_s v \rangle - \langle \nabla_s \tilde{u}, \nabla_s \tilde{u} \rangle) + \int_{\mathbb{R}^2} (W(\tilde{u} + v) - W(\tilde{u})) ds, \quad \xi \in \mathbb{R}^{n-2}.$$

With this definition we can represent the energy  $J_{\mathbb{R}^2 \times O}(v)$  of a generic map  $v \in W_G^{1,2}(\mathbb{R}^2 \times O; \mathbb{R}^m)$  in the *polar* form

$$(3.8) \quad J_{\mathbb{R}^2 \times O}(v) = \int_O \frac{1}{2} (|\nabla_\xi q^v|^2 + (q^v)^2 \sum_j \langle \nu_{\xi_j}^v, \nu_{\xi_j}^v \rangle) + \mathbf{E}(q^v \nu^v) d\xi,$$

where  $\langle \cdot, \cdot \rangle$  denotes the standard inner product in  $L^2(\mathbb{R}^2; \mathbb{R}^m)$  and  $q^v$  and  $\nu^v$  are defined by

$$(3.9) \quad \begin{aligned} q^v(\xi) &= \|v(\cdot, \xi) - \tilde{u}\|_{L^2(\mathbb{R}^2; \mathbb{R}^m)}, \quad \text{for } \xi \in O \\ \nu^v(s, \xi) &= \frac{v(s, \xi) - \tilde{u}(s)}{q^v(\xi)}, \quad \text{if } q^v(\xi) > 0. \end{aligned}$$

From and assumptions  $\mathbf{H}'_5$  and  $\mathbf{H}'_5$ , arguing exactly as in the proof of Lemma 2.1 we prove

**Lemma 3.2.**  $\mathbf{H}'_5$  and  $\mathbf{H}'_5$ . *Then there exist  $q^\circ > 0$  and  $c > 0$  such that*

$$(3.10) \quad \left\{ \begin{array}{l} D_{qq} \mathbf{E}(q\nu) \geq c^2, \quad \text{for } q \in [0, q^\circ] \cap [0, q_\nu], \nu \in \mathbb{S}, \\ \mathbf{E}(q\nu) \geq \mathbf{E}(q^\circ \nu), \quad \text{for } q^\circ \leq q \leq q_\nu, \nu \in \mathbb{S}, \\ \mathbf{E}(q\nu) \geq \tilde{\mathbf{E}}(p, q, \nu) := \mathbf{E}(p\nu) + D_q \mathbf{E}(p\nu)(q - p), \\ \quad \text{for } 0 \leq p < q \leq q_\nu \leq q^\circ, \nu \in \mathbb{S}, \\ D_p \tilde{\mathbf{E}}(p, q, \nu) \geq 0, \quad \text{for } 0 \leq p < q \leq q_\nu \leq q^\circ, \nu \in \mathbb{S}. \end{array} \right.$$

#### Step 4

Based on this lemma and on the polar representation of the energy (3.8) we can follow step by step the arguments in Sec. 2 to establish the analogous of Proposition 2.8. Actually the argument simplifies since by Lemma 3.1 we can work directly in  $\mathbb{R}^2 \times O$  rather than in bounded cylinders as in Sec. 2. For example the analogous of Lemma 2.3 is not needed. In conclusion, by arguing as in Sec. 2, we prove that, given  $q \in (0, q^\circ]$ , there is  $r(q) > 0$  such that

$$(3.11) \quad \mathbb{R}^2 \times B_{r(q)}(\xi) \subset \mathbb{R}^n \Rightarrow q^u(\xi) = \|u(\cdot, \xi) \tilde{u}\|_{L^2(\mathbb{R}^2; \mathbb{R}^m)} < q,$$

where  $B_{r(q)}(\xi) \subset \mathbb{R}^{n-2}$  is the ball of center  $\xi$  and radius  $r(q)$ . Since the condition on the l.h.s. of (3.11) is trivially satisfied for all  $\xi \in \mathbb{R}^{n-2}$  and for all  $q \in (0, q^\circ]$  we have

$$u(s, \xi) = \tilde{u}(s), \quad \text{for } s \in \mathbb{R}^2, \xi \in \mathbb{R}^{n-2}$$

which concludes the proof.

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Department of Mathematics, University of Athens, Panepistemiopolis, 15784 Athens, Greece; e-mail: [nalikako@math.uoa.gr](mailto:nalikako@math.uoa.gr)

Università degli Studi dell'Aquila, Via Vetoio, 67010 Coppito, L'Aquila, Italy; e-mail: [fusco@univaq.it](mailto:fusco@univaq.it)